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Catastrophe Risk Bonds with Applications to California Earthquakes

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Abstract Catastrophe (CAT) risk bonds provide a solid mechanism for direct transfer of extreme events (hazards) into the financial market. The last two decades, the insurance companies are seeking for more adequate and liquidity funds since the losses from the climate changes and natural disasters have been increased significantly over the time. In this paper we have two aims: first to study the pricing process of CAT bonds for the structure of n financial and m catastrophe independent risks generalizing the ideas of the paper by Cox and Perdersen [12]. Second to illustrate the applicability of our results, the historical data of the maximum magnitude of California earthquakes using the tool of the Extreme Value Theory are considered, together with latitude, longitude and depth of the corresponding earthquakes. More precisely, under the standard arbitrage-free valuation framework and its connection with the theory of equilibrium pricing, a multi-variable probabilistic model is developed. Additionally, an appropriate model for the term structure of interest rate and for inflation rate dynamics, and a stochastic process of coupon rate have been evaluated. Finally, based on the analysis for the aforementioned catastrophe and financial market risks, we are able to use equilibrium pricing theory and extend the work provided by Zimbidis *et al.* [61] to find a certain value price for the CAT California earthquake bond by Monte-Carlo Simulation.

Keywords. CAT Risk Bonds; Extreme Value Theory; Equilibrium pricing; Monte-Carlo Simulations.

1 Introduction

CAT bonds directly transfer the financial consequences of catastrophe events which cause severe losses from issuers to investors, in contract to cover the possible huge liabilities through traditional reinsurance providers or governmental budgets. The insurance companies are seeking to alleviate part of the risk through a more adequate

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and liquidity fund since the losses from the climate changes and natural catastrophes increased significantly during the past two decades.

The CAT bonds are inherently risky non-indemnity-based multi period deals, which pay a coupon to investors at the end of each period and a final principal payment at the maturity date given no pre-specified degree catastrophe occurred. The default of a CAT bond occurs when a major catastrophe hit the secured region before the expiry date, investors will receive no coupon payment or only pay back part of the principle. Generally, CAT bonds carry a 3 to 5 years maturity at issuance, and a floating coupon of Libor plus a premium at rate between 2% and 20% [14], [56].

The first experimental transaction was completed in the mid-1990s after the Hurricane Andrew with losses of USD 19.6bn and the Northridge earthquake with losses of USD 14.9bn by a number of specialized catastrophe-oriented insurance and reinsurance companies in USA, such as AIG, Hannover Re, St. Paul Re, and USAA [44]. The market grew rapidly from just over USD 0.6bn to over USD 2bn per year following 9-11 in 2001, and the claim of insured losses boom to USD 116bn in 2011 [55]. Specialists are seeking to make reinsurance market more risk-bearing.

According to Global Insurance Market Report (GIMAR), the catastrophes claimed over USD 350bn economic losses in the calendar year of 2011 [29]. By way of comparison, insured losses were approximately USD 116bn in the same year which shows there was still a huge gap between the total economic losses and secured losses. The most expensive insured natural disaster in 2011 was the 9.0 magnitude earthquake in Japan with loss of USD 210bn while USD 35bn insured, followed by the New Zealand earthquake with insurance claims of USD 12bn covering 80% economic losses [55]. Low insurance penetration rates leaves the individuals, companies, and governments to shoulder the remaining financial losses to catastrophic events. Based on the National Earthquake Information Center (U.S.) report, there are 12,000 to 14,000 recorded earthquakes annually throughout the world, while each year in California, two or three earthquakes have the magnitude 5.5 and higher which are large enough to cause moderate damage [53]. Although infrequent, earthquakes in addition to the side effects, such as landslides, surface fault ruptures, liquefaction, after shock fires and tsunamis cause huge potential loss of life, injury or property damage. Additionally, California Geological Survey reports more than 70% of residents live within the area where significant earthquakes could occur in the next 50 years based on the slip rates in geological time [53]. Therefore, the potential enormous financial demands on the insurance and reinsurance businesses make it realistic to introduce a securitization mechanic to prevent vulnerable individuals from catastrophic events such as earthquakes. In 2007, Swiss Re [54] launched a set of catastrophe bond performance indices, named Swiss Re Cat Bond Indices and published on Bloomberg which increased the transparency of CAT bond return.

The most significant issue of the catastrophe related financial instruments is to evaluate the present value for CAT earthquake bonds, see for instance Briys [7]. However, the pricing of CAT bonds requires an incomplete markets framework because the catastrophe risks can not be replicated by a portfolio of primitive securities, see Harrison and Kreps [30], Cox *et al.* [11], Cox and Pedersen [12], and Vaugirard [57]. Now, in the case of an incomplete market, there is no universal pricing theory which successfully covers all the aspects, such as specification of hedging strategies

and robustness of prices. For example, Wang [58] addresses market incompleteness by Wang transform, and this approach is adopted by Lin and Cox [36]-[37], Pelsser [46] and Galeotti *et al.* [25]. Alternative common technique used in the literature of incomplete market setting is the principle of equivalent utility in order to obtain the indifferent pricing. Young [59] calculates the price of a contingent claim under a stochastic interest rate for an exponential utility function. An extension of [59] is the paper by Egami and Young [16] which proposes a more complex payment structure. Cox and Pedersen [12] involve time repeatable representative agent utility, and a similar setting is used in multiple-event CAT bond for the first time by Reshetar [48].

Several important alternative pricing mechanisms have been developed for catastrophe-linked securities pricing models in different markets. Froot and Posner [22]-[23] derive an equilibrium pricing model for uncertain parameters of multi-events risks. In addition, a CAT bond model based on consumption is applied by Dieckmann [15], while Zhu [60] details the premium spread by using intertemporal equilibrium framework, and thereafter, Braun [6] analyzes the premium by OLS regressions with robust standard errors. Moreover, Young [59] claims that in order to price a financial product, default risk is completely eliminated or hedged by a previsible portfolio that has the same payoffs as the product and employs exponential utility line. Also, Föllmer and Schweizer [20] introduces the minimal martingale measure for option pricing, Schweizer [50] uses variance optimal martingale measure. Other possible equivalent martingale measures are Esscher martingale measure [27] and [8], and minimal entropy martingale measure, Miyahara [40]-[41] and Frittelli [21]. Lin *et al.* [35] applied a Markov-modulated Poisson process for the catastrophes occurrence using the similar approach as Vaugirard [57], and Nowak and Romaniuk [43] price the CAT bond focusing on the dynamic of spot interest rate. It is important to notice that Vaugirard [57] developed a simple arbitrage approach to value catastrophe risk insurance-linked securities at the first time, notwithstanding a non-traded underlying framework.

Another stand of the literature focuses on pricing via discrete or continuous time. For instance, in the discrete time framework, Cox and Pedersen [12] based on a model of the term structure of interest rates and a probability structure for the catastrophe risk assuming that the agent uses the utility function to make choices about consumption streams, and apply to Morgan Stanley Bond, Winterthur's Bonds, USAA's Bonds and Winterthur-style Bonds. Zimbidis *et al.* [61] also adopt the discrete time in bond pricing using the equilibrium pricing theory with dynamic interest rates. For the continuous time model, Pérez-Frutoso [47] develops a CAT bond with index triggers. Moreover, Loubergé *et al.* [38] employed a compound Poisson process for binomial interest rate. An extension of compound doubly stochastic Poisson process is delivered by Burnecki and Kukla [9] and Albrecher *et al.* [1].

In this paper, the Cox and Pedersen [12] approach is considered into a multi catastrophic and multi financial risks model for discrete time period. We assume (i) an incomplete and no arbitrage framework, (ii) the independence of all the possible risks, (iii) aggregate consumptions depends on financial and catastrophic risks. We provide a quick overview of the probability structure for the model in section 2 and design a price model for earthquake CAT bond by Equilibrium Pricing Theory. Fortunately enough, the fact that the catastrophe risks are uncorrelated with movements of the

financial risks make the problem much simpler. In section 3, we will specify a one period and a multi-period CAT bond price formulas, and analyse the term structures or the distributions of the risk variables relative to the bond. The distribution of annual maximum magnitude of earthquakes in California is estimated relying on the tool of Extreme Value Theory. The LIBOR rate dynamics is assumed to be a CIR model and the interest and inflation rate follow ARIMA processes. In section 4, detail numerical examples of a 1-year and a 5-year period CAT bond will be illustrated, and we will obtain the prices using Monte Carlo Simulation and Iterative Stochastic Equations. Moreover, we derive the density plot of the price to check the validity of our valuation. Finally in Section 5 we provide a discussion on the results and suggest the future directions.

2 Modeling CAT Bonds

2.1 Model description and preliminaries

In this section we give the preliminary presentation of the CAT bond structure. Generalizing the idea of Cox and Pedersen [12], we design a CAT bond to combine n financial market variables and m catastrophe risk variables. The model set up requires some probabilistic structure which is given as follows.

Let us assume that we are trading catastrophe risk bonds in an investment market that is arbitrage-free. The time of the catastrophe(s) is independent of the term structure(s) under the relevant probability measure. We assume that we have n financial risk variables, each one modeled on the probability triples $(\Omega_{1,i}, \mathcal{F}^{(1,i)}, \mathbb{P}_{1,i})$ for $i = 1, 2, \dots, n$. Let $T < \infty$ be the maturity time of the trading interval. Based on this, each sample space $\Omega_{1,i}$, $i = 1, 2, \dots, n$, is taken to be finite and represents all of the paths that the i^{th} financial variable can take over the times $k = 0, 1, \dots, T$. Also, let $\mathcal{F}_k^{(1,i)}$ be the σ -algebras of $\Omega_{1,i}$ representing the investment information (e.g. past security prices) available to the market at time k , where $k = 0, 1, \dots, T$. Then, by definition, each filtration $\mathcal{F}^{(1,i)}$, $i = 1, 2, \dots, n$, represents all the investment information up to time T . Thus, each probability measure $\mathbb{P}_{1,i}$ is defined for all the events belonging to the $\mathcal{F}_k^{(1,i)}$ σ -algebra, $k \leq T$. Note that the measures $\mathbb{P}_{1,i}$ do not necessarily have the same distribution as each other.

On the other hand, we consider m catastrophe risk variables, which are modeled on probability triples $(\Omega_{2,j}, \mathcal{F}^{(2,j)}, \mathbb{P}_{2,j})$, where, similar to before, $\mathcal{F}_k^{(2,j)}$ are the σ -algebras of $\Omega_{2,j}$ representing the risk information available in time k , $k = 0, 1, \dots, T$ and $\mathbb{P}_{2,j}$, $j = 1, 2, \dots, m$, are the probability measures governing the catastrophe structure (not necessarily having the same distribution). The filtrations $\mathcal{F}^{(2,j)}$ are indexed by the same times $k = 0, 1, \dots, T$, as previously. In practice, $\mathbb{P}_{2,j}$, $j = 1, 2, \dots, m$, are the probability measures used to compute the probability of every catastrophic event.

Based on the above sample spaces, we can construct the sample space of the full model, given by

$$\Omega = \left(\Omega_{1,1} \times \Omega_{1,2} \times \dots \times \Omega_{1,n} \right) \times \left(\Omega_{2,1} \times \Omega_{2,2} \times \dots \times \Omega_{2,m} \right),$$

such that a typical event of the full model sample space is of the form $\omega = (\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m})$, where $\tilde{\omega}_{\kappa,\ell} = (w_{\kappa,1}, w_{\kappa,2}, \dots, w_{\kappa,\ell})$, $\kappa = 1, 2$, $\ell = n, m$, such that $w_{1,i} \in \Omega_{1,i}$, $i = 1, 2, \dots, n$ and $w_{2,j} \in \Omega_{2,j}$, $j = 1, 2, \dots, m$. The ω element can be interpreted as the joint events of the n different financial risk variables and the m catastrophic risk variables.

Assuming that the events $w_{\kappa,1}, w_{\kappa,2}, \dots, w_{\kappa,\ell}$, $\kappa = 1, 2$, $\ell = n, m$, are pairwise independent, then probability measure on the sample space Ω is given by the natural product measure structure

$$\mathbb{P}(\omega) = \prod_{i=1}^n \mathbb{P}_{1,i}(\omega_{1,i}) \cdot \prod_{j=1}^m \mathbb{P}_{2,j}(\omega_{2,j}), \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m.$$

This assumption implies the independence of the events that rely on any of the $n + m$ risk variables.

In addition, the natural filtration produced by the σ -algebras of Ω is denoted by \mathcal{F} and given by

$$\mathcal{F}_k = \left(\mathcal{F}_k^{(1,1)} \times \mathcal{F}_k^{(1,2)} \times \dots \times \mathcal{F}_k^{(1,n)} \right) \times \left(\mathcal{F}_k^{(2,1)} \times \mathcal{F}_k^{(2,2)} \times \dots \times \mathcal{F}_k^{(2,m)} \right),$$

for $k = 0, 1, \dots, T$. Thus, with all the elements defined above, $(\Omega, \mathcal{F}, \mathbb{P})$ constructs a probability triple for the full model.

Based on the above definitions, we are able to define random variables for the full model that will depend either on financial variables or on catastrophic variables. For this we need to define the following filtrations, namely $\mathcal{A}^{(1)} = \{\mathcal{A}_1^{(1)} \subseteq \dots \subseteq \mathcal{A}_T^{(1)}\}$, $\mathcal{A}^{(1,i)} = \{\mathcal{A}_1^{(1,i)} \subseteq \dots \subseteq \mathcal{A}_T^{(1,i)}\}$, for $i = 1, \dots, n$, and similarly $\mathcal{A}^{(2)} = \{\mathcal{A}_1^{(2)} \subseteq \dots \subseteq \mathcal{A}_T^{(2)}\}$, $\mathcal{A}^{(2,j)} = \{\mathcal{A}_1^{(2,j)} \subseteq \dots \subseteq \mathcal{A}_T^{(2,j)}\}$, for $j = 1, \dots, m$, generated from the following σ -algebras,

$$\begin{aligned} \mathcal{A}_k^{(1)} &= \mathcal{F}_k^{(1,1)} \times \dots \times \mathcal{F}_k^{(1,n)} \times \{\emptyset, \Omega_{2,1}, \dots, \Omega_{2,m}\}, \\ \mathcal{A}_k^{(1,i)} &= \mathcal{F}_k^{(1,i)} \times \{\emptyset, \Omega_{2,1}, \dots, \Omega_{2,m}\}, \quad i = 1, \dots, n, \\ \mathcal{A}_k^{(2)} &= \{\emptyset, \Omega_{1,1}, \dots, \Omega_{1,n}\} \times \mathcal{F}_k^{(2,1)} \times \dots \times \mathcal{F}_k^{(2,m)}, \\ \mathcal{A}_k^{(2,j)} &= \{\emptyset, \Omega_{1,1}, \dots, \Omega_{1,n}\} \times \mathcal{F}_k^{(2,j)}, \quad j = 1, \dots, m, \end{aligned}$$

for $k = 1, \dots, T$. An $\mathcal{A}_T^{(\kappa)}$ measurable random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ (or an $\mathcal{A}^{(\kappa)}$ adapted stochastic process Y) is depend on financial risk variables ($\kappa = 1$) or catastrophic risk variables ($\kappa = 2$). Then, let financial events $\alpha_{1,i} \in \mathcal{A}_T^{(1,i)}$ and catastrophic events $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$. Therefore, $\alpha_{1,i} = A_{1,i} \times \Omega_{2,1} \times \dots \times \Omega_{2,m}$ and $\alpha_{2,j} = \Omega_{1,1} \times \dots \times \Omega_{1,n} \times A_{2,j}$, for some $A_{\kappa,\ell} \in \mathcal{F}_T^{(\kappa,\ell)}$, $\kappa = 1, 2$, $\ell = i, j$. We need the independent notion of $\mathcal{A}_T^{(\kappa,\ell)}$ due to the fact that we cannot refer to $\mathcal{F}_T^{(\kappa,\ell)}$ as being independent under \mathbb{P} , since each of the $\mathcal{F}_T^{(\kappa,\ell)}$, does not contain events defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 1 For $i = 1, \dots, n$, and $j = 1, \dots, m$ the σ -algebras $\mathcal{A}_T^{(1,i)}$ and $\mathcal{A}_T^{(2,j)}$ are independent under the probability measure \mathbb{P} .

Proof For $i = 1, \dots, n$ and $j = 1, \dots, m$, let $a_{1,i} \in \mathcal{A}_T^{(1,i)}$ and $a_{2,j} \in \mathcal{A}_T^{(2,j)}$. Then, $a_{1,i} = A_{1,i} \times \Omega_{2,1} \times \dots \times \Omega_{2,m}$, for some $A_{1,i} \in \mathcal{F}_k^{(1,i)}$ and $a_{2,j} = \Omega_{1,1} \times \dots \times \Omega_{1,n} \times A_{2,j}$, for some $A_{2,j} \in \mathcal{F}_k^{(2,j)}$, we have that

$$\begin{aligned}
& \mathbb{P} \left[\left(\bigcap_{i=1}^n \alpha_{1,i} \right) \cap \left(\bigcap_{j=1}^m \alpha_{2,j} \right) \right] = \mathbb{P} \left(A_{1,1} \times \dots \times A_{1,n} \times A_{2,1} \times \dots \times A_{2,m} \right) \\
&= \prod_{i=1}^n \mathbb{P}_{1,i}(A_{1,i}) \cdot \prod_{j=1}^m \mathbb{P}_{2,j}(A_{2,j}) \\
&= \prod_{i=1}^n \mathbb{P}_{1,i}(A_{1,i}) \prod_{j=1}^m \mathbb{P}_{2,j}(\Omega_{2,j}) \cdot \prod_{i=1}^n \mathbb{P}_{1,i}(\Omega_{1,i}) \prod_{j=1}^m \mathbb{P}_{2,j}(A_{2,j}) \\
&= \prod_{i=1}^n \mathbb{P} \left(A_{1,i} \times \Omega_{2,1} \times \dots \times \Omega_{2,m} \right) \cdot \prod_{j=1}^m \mathbb{P} \left(\Omega_{1,1} \times \dots \times \Omega_{1,n} \times A_{2,j} \right) \\
&= \prod_{i=1}^n \mathbb{P}(\alpha_{1,i}) \cdot \prod_{j=1}^m \mathbb{P}(\alpha_{2,j}).
\end{aligned}$$

2.2 The valuation framework

In this section, we show how one can implement valuation under the full model by choosing the equivalent measure.

The presence of catastrophic risks lead us to consider an incomplete market. In this case, there is no universal theory that addresses all aspects of pricing. Various alternative mechanisms have been developed, and they have tried to price the uncertain cash flow streams that appear. In this paper, similar to Cox and Pedersen [12] and Magill and Quinzii [39], we adopt the setting of the representative agent in order to price the uncertain cash flow streams.

Let us suppose that we are in a T -period economy in which agents can make choices and consume during each period. The agent makes choices about his future consumption, represented by the stochastic process $\{c(k); k = 0, 1, \dots, T\}$. Also we introduce the aggregate consumption process, which may be thought of as the total consumption available in the economy (for all agents) at each point in time and in each state of the world. The aggregate consumption stochastic process is denoted by $\{C^*(k); k = 0, 1, \dots, T\}$. Both the above processes adapt to the filtration of the full model. Only the first choice is known with certainty at time $k = 0$.

In many applications it is customarily assumed that the representative agent's utility is time-additive, separable and differentiable as in Cox and Pedersen [12]¹. An immediate consequence of this assumption is that there are utility functions u_0, u_1, \dots, u_T such that the agent's utility for a consumption process $\{c(k); k =$

¹ For more details about the representative agent theory in the incomplete market we refer to Huang and Litzenberger [31], Karatzas [32], Embrechts and Meister [17].

$0, 1, \dots, T\}$ is given by

$$\mathbb{E}^{\mathbb{P}} \left[\sum_{k=0}^T u_k(c(k)) \right].$$

Then, from the theory of the representative agent, the price which is denoted by $V(d)$ of a generic future cash flow process $\{d(k); k = 1, 2, \dots, T\}$ at time 0 is given by

$$V(d) = \mathbb{E}^{\mathbb{P}} \left[\sum_{k=1}^T \frac{u'_k(C^*(k))}{u'_0(C^*(0))} d(k) \right]. \quad (1)$$

More generally, the general inter-temporal valuation of future cash-flow process $\{d(k); k = p+1, p+2, \dots, T\}$, is given by the conditional expectation

$$\mathbb{E}^{\mathbb{P}} \left[\sum_{k=p+1}^T \frac{u'_k(C^*(k))}{u'_p(C^*(p))} d(k) \middle| \mathcal{F}_p \right]. \quad (2)$$

Note from the pricing relations (1)-(2) that the evaluation of the price at time zero (or p) heavily depends on the aggregate consumption process and the choice of the utility functions. The non-observable form of the consumption process and the utility functions will be removed by relating the pricing relation (1) to the valuation measure approach of arbitrage-free pricing. For this purpose we need first to define the one-period financial market influence rates implicit in the representative agent pricing model.

For $i = 1, 2, \dots, n$, let $\{r_i(k); k = 0, 1, 2, \dots, T-1\}$ be the one period financial market influence rates. Then, the one-period financial market influence rates could be define through the conditional expectation

$$\prod_{i=1}^n \frac{1}{1+r_i(k)} = \frac{1}{u'_k(C^*(k))} \mathbb{E}^{\mathbb{P}} [u'_{k+1}(C^*(k+1)) | \mathcal{F}_p], \quad k = 0, 1, 2, \dots, T-1. \quad (3)$$

Now, following the same line of logic as in Cox and Pedersen [12], a new probability measure $\mathbb{Q}(\cdot)$ is defined in terms of $\mathbb{P}(\cdot)$ and a positive random variable, called the Randon-Nikodym derivative of \mathbb{Q} with respect to \mathbb{P} . Under this new measure, all prices are discounted with respect to the product of one-period financial market influence rates $r_i(k)$. To define the Randon-Nikodym derivative, let

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{i=1}^n \prod_{s=0}^{T-1} [1+r_i(s)] \cdot \frac{u'_T(C^*(T))}{u'_0(C^*(0))}. \quad (4)$$

Note that this new random variable is measurable with respect to \mathcal{F}_T . Also, clearly we need to ensure that $\mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] = 1$ where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is defined by the above equation. For this we need Lemma 2 below. In order to prove our lemma, we need to define the following stochastic processes.

Firstly, for notation simplicity, we denote by $B(k) = \prod_{i=1}^n \prod_{s=0}^{k-1} [1+r_i(s)]$, $k = 1, 2, \dots, T$, (with $B(0) = 1$) the one-period financial market discount rates.

Then, we are able to define the stochastic processes $\{\xi(k); k = 0, 1, \dots, T\}$ and $\{\zeta(k); k = 0, 1, \dots, T\}$ as

$$\xi(k) = \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_k \right] = \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_k},$$

and

$$\zeta(k) = B(k) \cdot \frac{u'_k(C^*(k))}{u'_0(C^*(0))}, \quad k = 1, \dots, T,$$

respectively. Since $B(0) = 1$, it is easy to check that $\zeta(0) = 1$. Also note that from Eq. (4) it holds that

$$\zeta(T) = \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{F}_T.$$

Lemma 2 *The process $\{\zeta(k); k = 0, 1, 2, \dots, T\}$ is a \mathbb{P} -martingale on the filtration \mathcal{F} , and $\zeta(k) = \xi(k)$, for $k = 0, 1, 2, \dots, T$.*

Proof Similar with Lemma B.1, Cox and Pedersen [12], so it is omitted.

Remark 1 Immediate consequence of the Lemma 2 is that

$$1 = \mathbb{E}^{\mathbb{P}} [\zeta(0)] = \mathbb{E}^{\mathbb{P}} [\zeta(T)] = \mathbb{E}^{\mathbb{P}} [\xi(T)] = \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right],$$

which ensures that the Radon-Nikodym derivative in Eq. (4) does indeed define a new probability measure.

Intuitively the probability measure $\mathbb{Q}(\cdot)$ is equivalent to a knowledge of the representative investor's utility function and the aggregate consumption process. Thus, we may use this new measure in order to find an equivalent expression to Eq. (1), for the evaluation of a generic cash flow at time 0. Hence, using similar arguments as in the Theorem B.1 of Cox and Pedersen [12] we have the following theorem.

Theorem 1 *Under the assumptions of the representative agent pricing model, the price of a generic future cash flow process $\{d(k); k = 1, 2, \dots, T\}$ at time 0 is given by*

$$V(d) = \mathbb{E}^{\mathbb{Q}} \left[\sum_{k=1}^T \frac{1}{\prod_{i=1}^n \prod_{s=0}^{k-1} [1 + r_i(s)]} d(k) \right] = \mathbb{E}^{\mathbb{Q}} \left[\sum_{k=1}^T \frac{1}{B(k)} d(k) \right]. \quad (5)$$

Remark 2 As the nature of the incomplete market, there is no unique interpretation of the prices that we assign to the catastrophe risk bonds unless introduce the probability distribution of the catastrophe risk. This problem is often met in any model that is used to attach a price to catastrophe risk bonds. For more details see Cox and Pedersen [12].

Similarly, using arguments as in the Theorem B.2 of Cox and Pedersen [12], the general inter-temporal valuation of a future cash flow, given in Eq. (2) can be expressed in terms of the equivalent measure $\mathbb{Q}(\cdot)$.

Theorem 2 *Under the assumptions of the representative agent pricing model, the price of a generic future cash flow process $\{d(k); k = p + 1, p + 2, \dots, T\}$ is given by*

$$\mathbb{E}^{\mathbb{P}} \left[\sum_{k=p+1}^T \frac{u'_k(C^*(k))}{u'_p(C^*(p))} d(k) \middle| \mathcal{F}_p \right] = \mathbb{E}^{\mathbb{Q}} \left[\sum_{k=p+1}^T \frac{B(p)}{B(k)} d(k) \middle| \mathcal{F}_p \right].$$

For the purpose of analyzing CAT bonds, from this point onwards, we will assume that the aggregate consumption depends only on the n financial risk variables, given as $C^*(\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m}; k) = C^*(\tilde{\omega}_{1,n}; k)$, for $\omega \equiv (\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m}) \in \Omega$, then, C^* is $\mathcal{A}^{(1)}$ adapted. It is quite a natural approximation since the global economic circumstances in terms of exchange and production are not strongly related to the localized catastrophes [12].

The aggregate consumption process depends only on all financial risks information available at time k , given the structure at time 0 is known. Therefore, we can rewrite any $\mathcal{A}_k^{(1)}$ measurable random variable which conditioning on the full information in \mathcal{F}_k as conditioning on $\mathcal{A}_k^{(1)}$.

Lemma 3 *Under the assumption that C^* is $\mathcal{A}^{(1)}$ adapted, for any random variable X that is $\mathcal{A}_T^{(2)}$ measurable we have*

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X].$$

Epecially, for any catastrophic events $\alpha_{2,j}$, $j = 1, 2, \dots, m$, that are $\mathcal{A}_T^{(2,j)}$ measurable, it holds that

$$\mathbb{Q}(\bigcap_{j=1}^m (\alpha_{2,j})) = \mathbb{P}(\bigcap_{j=1}^m (\alpha_{2,j})) = \prod_{j=1}^m \mathbb{P}_{2,j}(A_{2,j}). \quad (6)$$

Proof Notice that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in Eq. (4) is $\mathcal{A}_T^{(1)}$ measurable according to the fact that C^* and $B(T)$ are $\mathcal{A}^{(1)}$ adapted. Since the r.v.s X and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ are $\mathcal{A}_T^{(2)}$ measurable and independent under \mathbb{P} , together with Lemma 3.2.5 of Shreve [51], we can have

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}} \left[X \frac{d\mathbb{Q}}{d\mathbb{P}} \right] = \mathbb{E}^{\mathbb{P}}[X] \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] = \mathbb{E}^{\mathbb{P}}[X] \cdot 1 = \mathbb{E}^{\mathbb{P}}[X].$$

Moreover, define

$$X = \prod_{j=1}^m \mathbb{1}_{\alpha_{2,j}} = \mathbb{1}_{\bigcap_{j=1}^m \alpha_{2,j}},$$

where $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$, $j = 1, 2, \dots, m$, substitute in Eq. (6), we have

$$\begin{aligned} \mathbb{Q}\left(\bigcap_{j=1}^m (\alpha_{2,j})\right) &= \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\bigcap_{j=1}^m \alpha_{2,j}} \right] = \mathbb{E}^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{j=1}^m \alpha_{2,j}} \right] \\ &= \mathbb{P}\left(\bigcap_{j=1}^m (\alpha_{2,j})\right) = \mathbb{P} \left[\bigcap_{j=1}^m \{ \Omega_{1,1} \times \dots \times \Omega_{1,m} \times A_{2,j} \} \right] \\ &= \prod_{j=1}^m \left[\left(\prod_{i=1}^n \mathbb{P}(\Omega_{1,i}) \right) \mathbb{P}(A_{2,j}) \right] = \prod_{j=1}^m \mathbb{P}_{2,j}(A_{2,j}). \end{aligned}$$

Remark 3 Intuitively, under the measure $\mathbb{P}(\cdot)$, and under the assumption that C^* depends only on the financial risks variables, the catastrophic events $\alpha_{2,j}$ which depend on the j th catastrophic risk, $j = 1, \dots, m$, are independent.

In order to implement Theorems 1 and 2, we need events to be independent from each other, for those that depend only on financial risks and depend only on the catastrophic risks, under the measure of \mathbb{Q} .

Lemma 4 Under the assumption that C^* is $\mathcal{A}^{(1)}$ adapted, the σ -algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent under \mathbb{Q} .

Proof Let $\alpha_{1,i} \in \mathcal{A}_T^{(1,i)}$, and similarly $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$. Applying the Lemma 3.2.5 of Shreve [51], we have

$$\begin{aligned} \mathbb{Q}\left(\left(\bigcap_{i=1}^n \alpha_{1,i}\right) \cap \left(\bigcap_{j=1}^m \alpha_{2,j}\right)\right) &= \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\bigcap_{i=1}^n \alpha_{1,i}} \mathbb{1}_{\bigcap_{j=1}^m \alpha_{2,j}} \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{i=1}^n \alpha_{1,i}} \mathbb{1}_{\bigcap_{j=1}^m \alpha_{2,j}} \frac{d\mathbb{Q}}{d\mathbb{P}} \right]. \end{aligned}$$

Since $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in Eq. (4) is $\mathcal{A}_T^{(1)}$ measurable, therefore

$$\mathbb{1}_{\bigcap_{i=1}^n \alpha_{1,i}} \frac{d\mathbb{Q}}{d\mathbb{P}} \quad \text{and} \quad \mathbb{1}_{\bigcap_{j=1}^m \alpha_{2,j}}$$

are independent under \mathbb{P} . Consequently,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{i=1}^n \alpha_{1,i}} \mathbb{1}_{\bigcap_{j=1}^m \alpha_{2,j}} \frac{d\mathbb{Q}}{d\mathbb{P}} \right] &= \mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{i=1}^n \alpha_{1,i}} \frac{d\mathbb{Q}}{d\mathbb{P}} \right] \mathbb{E}^{\mathbb{P}} \left[\mathbb{1}_{\bigcap_{j=1}^m \alpha_{2,j}} \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\mathbb{1}_{\bigcap_{i=1}^n \alpha_{1,i}} \right] \mathbb{P} \left[\bigcap_{j=1}^m \alpha_{2,j} \right] \\ &= \mathbb{Q} \left[\bigcap_{i=1}^n \alpha_{1,i} \right] \prod_{j=1}^m \mathbb{P}_{2,j}[\alpha_{2,j}]. \end{aligned}$$

Refer back to lemma 3, we can have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\cap_{i=1}^n \alpha_{1,i}} \mathbb{1}_{\cap_{j=1}^m \alpha_{2,j}} \frac{d\mathbb{Q}}{d\mathbb{P}}\right] &= \mathbb{Q}\left[\bigcap_{i=1}^n \alpha_{1,i}\right] \prod_{j=1}^m \mathbb{P}_{2,j}[\alpha_{2,j}] \\ &= \mathbb{Q}\left[\bigcap_{i=1}^n \alpha_{1,i}\right] \mathbb{Q}\left[\bigcap_{j=1}^m \alpha_{2,j}\right],\end{aligned}$$

therefore, we conclude that under \mathbb{Q} the σ -algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent.

As a direct implication of Lemma 3 and 4, the current value of cash flows X depending on m catastrophic risks variables have the simple form as below. For notation simplicity, we denote the current value of a face amount 1, non-defaultable zero-coupon bond maturing at time k as $P(k) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(k)}\right]$.

Corollary 1 *The current value of an $\mathcal{A}_k^{(2)}$ measurable cash flow X paid at time k , is given by*

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(k)}X\right] = P(k)\mathbb{E}^{\mathbb{P}}[X].$$

Under the discrete time framework, we can express the valuation measure as a product measure of the probability measures \mathbb{Q}_1 and $\mathbb{P}_{2,j}$,

$$\begin{aligned}\mathbb{Q}(\omega) &= \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)\mathbb{P}(\omega) \\ &= B(\omega; T) \frac{u'_T(C^*(\omega; T))}{u'_0(C^*(\omega; 0))} \mathbb{P}(\omega) \\ &= \prod_{s=0}^{T-1} \left[\prod_{i=1}^n [1 + r_i(\omega_{1,i}; s)] \right] \frac{u'_T(C^*(\tilde{\omega}_{1,n}; T))}{u'_0(C^*(\tilde{\omega}_{1,n}; 0))} \prod_{i=1}^n \mathbb{P}_{1,i}(\omega_{1,i}) \prod_{j=1}^m \mathbb{P}_{2,j}(\omega_{2,j}) \\ &= \mathbb{Q}_1(\tilde{\omega}_{1,n}) \prod_{j=1}^m \mathbb{P}_{2,j}(\omega_{2,j}),\end{aligned}\tag{7}$$

where

$$\mathbb{Q}_1(\tilde{\omega}_{1,n}) = \prod_{s=0}^{T-1} \left[\prod_{i=1}^n [1 + r_i(\omega_{1,i}; s)] \right] \frac{u'_T(C^*(\tilde{\omega}_{1,n}; T))}{u'_0(C^*(\tilde{\omega}_{1,n}; 0))} \prod_{i=1}^n \mathbb{P}_{1,i}(\omega_{1,i}). \tag{8}$$

The probability measure in Eq. (8) is generated in terms of financial risks term structure, Pedersen [45]. It is practical to have Eq. (7) since empirical probabilities of the catastrophic events can be used for the probability measures $\mathbb{P}_{2,j}$, where $j = 1, \dots, m$. We will formalize the value for certain types of CAT bonds in Section 2.3.

2.3 Implication for Valuation

In this section, we will give the concrete form for pricing certain CAT bonds under the discrete time framework.

The valuation structure of CAT bonds can be further simplified due to the fact that the discount factors $B(k)$ in Eq. (5) are $\mathcal{A}_k^{(1)}$ measurable which depend only on financial influence risks variables. Let us consider a generic future cash flow process $d(\omega; k) = d(\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m}; k)$ depending on the n financial risk variables as well as the m catastrophic risk variables. In addition, we define an associated process of future cash flow process as

$$\bar{d}(k) = \mathbb{E}^{\mathbb{Q}}[d(k)|\mathcal{A}_k^{(1)}],$$

which is the conditional expectation over the loss distribution of catastrophic risks given fixed financial risks variables. The value of \bar{d} reflects the financial events by the filtration $\mathcal{A}^{(1)}$, thus $\bar{d}(k)$ is $\mathcal{A}_k^{(1)}$ measurable. We now reformulate Eq. (5) using the process \bar{d} , addition with $B(k)$ and $\bar{d}(k)$ are $\mathcal{A}_k^{(1)}$ measurable, as

$$V(d) = \mathbb{E}^{\mathbb{Q}} \left[\sum_{k=1}^T \frac{1}{B(k)} \bar{d}(k) \right] = \mathbb{E}^{\mathbb{Q}_1} \left[\sum_{k=1}^T \frac{1}{B(k)} \bar{d}(k) \right], \quad (9)$$

where \mathbb{Q}_1 is the valuation measure in terms of n financial risks variables given in Eq. (8). It is quite practical since we are able to use Eq. (9) to price the CAT bond by choosing arbitrage free financial risks term structure and calculating the expected cash flow conditionally on financial risk process.

However, in order to complete the valuation, we need to also verify the structure of the cash follow process. A direct deduction from Corollary 1 is the case that the CAT bond cash flows depend only on the m catastrophic risks variables.

Theorem 3 *For CAT bond cash flows which are $\mathcal{A}^{(2)}$ adapted,*

$$\bar{d}(k) = \mathbb{E}^{\mathbb{Q}} \left[d(k) | \mathcal{A}_k^{(1)} \right] = \mathbb{E}^{\mathbb{P}}[d(k)],$$

Eq. (9) can be simplified by the formula in Corollary 1 as

$$V(d) = \sum_{k=1}^T P(k) \mathbb{E}^{\mathbb{P}}[d(k)]. \quad (10)$$

The price of the CAT bond given in Eq. (9) and (10) as an extension of Cox and Pedersen [12] are the core results of this paper.

3 Application of the Results

In this section, we will apply the pricing model Eq. (9)-(10) to give a price for California earthquakes CAT bond. It is logical to consider interest rate, inflation rate, LIBOR rate, GDP level, and employment level as the sources of financial market influence

risk variables of CAT bonds. While, relevant catastrophe risk variables include magnitude of earthquake, the corresponding location and depth, outbreak of tsunami and fire, and ground condition, etc. In this application only, we are trying to introduce a model with three financial influence risks (Libor rate, real interest rate and inflation rate) and three catastrophe risks (magnitude, depth and place of the earthquakes).

We model one of the financial market risks, real interest rate, via a discrete process $\{r_k; k = 1, 2, \dots, T\}$ within $(\Omega_{1,1}, \mathcal{F}^{(1,1)}, \mathbb{P}_{1,1})$ which is equipped with the filtration $\mathcal{F}^{(1,1)}$. Similarly, we governing the inflation rate process $\{\pi_k; k = 1, 2, \dots, T\}$ within another complete probability triple $(\Omega_{1,2}, \mathcal{F}^{(1,2)}, \mathbb{P}_{1,2})$ which is equipped with the filtration $\mathcal{F}^{(1,2)}$. The final financial risk US LIBOR rate $R_k, k \in [0, T]$ is modeled within $(\Omega_{1,3}, \mathcal{F}^{(1,3)}, \mathbb{P}_{1,3})$ which is equipped with the filtration $\mathcal{F}^{(1,3)}$.

The catastrophic risks are modeled via three discrete time processes. We model the annual maximum magnitude earthquake in selected area in California by the process $\{M_k, k = 1, 2, \dots, T\}$ within the probability space $(\Omega_{2,1}, \mathcal{F}^{(2,1)}, \mathbb{P}_{2,1})$ which is equipped with the filtration $\mathcal{F}^{(2,1)}$. Thereafter, we model the corresponding depth process $\{D_k, k = 1, 2, \dots, T\}$ within $(\Omega_{2,2}, \mathcal{F}^{(2,2)}, \mathbb{P}_{2,2})$ which is equipped with the filtration $\mathcal{F}^{(2,2)}$, and region process $\{q_k, k = 1, 2, \dots, T\}$ within $(\Omega_{3,3}, \mathcal{F}^{(3,3)}, \mathbb{P}_{3,3})$ which is equipped with the filtration $\mathcal{F}^{(2,3)}$.

A one period and multi-period model will be developed and obtain the approximate prices for modeled CAT bond. This valuation is performed in three stages. The first stage is to specify the cash flows to the bondholder which are depended on the above risk variables. The next stage is to analysis each of the financial risk and catastrophe risk dynamics by assuming suitable distribution function and estimating parameters by historical data. The final stage is to generate sequences of discrete time process of future risks in order to simulate the expected future cash flows by Monte-Carlo simulation, and finally obtain the price of the CAT bonds in arbitrage-free framework.

3.1 One Period (basic) Model

In this subsection, a simple one-period model is formulated where the financial influence rate dynamics (real interest rate, inflation rate and LIBOR rate in this example) are restricted to constant values. Under the discrete framework of the analysis, we first define the following symbols and notations, i.e.

- K : is the face amount of the CAT bond.
- r : is the one period risk-free real interest rate (e.g. 1-year US Treasury securities rate).
- π : is the one period inflation rate (e.g. represented by all urban consumers Consumer Price Index (CPI)).
- R : is the deterministic coupon payment rate for the one year period given that a specified catastrophic event does not occur (e.g. 12-month US LIBOR rate at the date of issuing the bond in our case).
- e : is the extra premium loading for the earthquake risk (is normally positive considering investor averse from risk).

M : is the maximum magnitude level of the earthquake within all selected regions of California. To be more precise, $M = \max\{M_1, M_2\}$, and M_q is a random variable following the distribution obtained in Section 3.3.1, where $q = 1, 2$.

D_q : is the corresponding depth (Km) of the earthquake in the region q .

$V(d)$: is the price of the CAT bond at time of issuing depending on earthquakes occurring till maturity date.

$d(M, R, D_q)$: is the piecewise cash value of the CAT bond at time of maturity depending upon the catastrophe risk variables (level of magnitude M , depth D , and location q). Zimbidis *et al.* [61] give a similar expression for CAT bond cash flows which are depend on R and M . As an illustration, the structure of the cash value is given by

$$d(M, R, D_q) = \begin{cases} K \cdot (1 + f(R)), & M \in [0, M_a], \text{ with } \{D_q \leq D_a\} \text{ or } \{D_q > D_a\} \\ K \cdot (1 + g(R)), & M \in (M_a, M_b], \text{ with } \{D_q \leq D_b\} \text{ or } \{D_q > D_b\} \\ K \cdot (1 + h(R)), & M \in (M_b, M_c], \text{ with } \{D_q \leq D_c\} \text{ or } \{D_q > D_c\} \\ K, & M \in (M_c, M_d] \\ \phi(K), & M \in (M_d, M_e], \text{ with } \{D_q \leq D_d\} \text{ or } \{D_q > D_d\} \\ \gamma(K), & M \in (M_e, M_f], \text{ with } \{D_q \leq D_e\} \text{ or } \{D_q > D_e\} \\ \eta(K), & M \in (M_f, \infty) \end{cases},$$

where the trigger points M_a, M_b, \dots, M_f and D_a, D_b, \dots, D_e are pre-specified magnitude level and depth level, respectively, where $0 < M_a < M_b < \dots < M_f$ and $0 < D_a < D_b < \dots < D_e$. The selection of M_a, M_b, \dots, M_f affects securitization level of the bond, while individual company should balance between the profit and marketability by analyzing historical earthquake losses data. Finally, coupon payment functions $f(R)$, $g(R)$, $h(R)$, $\phi(K)$, $\gamma(K)$, and $\eta(K)$ are normally designed according to company policy. Here we illustrate a possible example,

$$\begin{aligned} f(R) &= \begin{cases} 2.6R \cdot \mathbb{1}_{\{D_1 \leq D_a\}} + 2.8R \cdot \mathbb{1}_{\{D_1 > D_a\}}, & \text{for } M_1 > M_2 \\ 2.9R \cdot \mathbb{1}_{\{D_2 \leq D_a\}} + 3R \cdot \mathbb{1}_{\{D_2 > D_a\}}, & \text{for } M_1 < M_2 \end{cases}, \\ g(R) &= \begin{cases} 1.6R \cdot \mathbb{1}_{\{D_1 \leq D_b\}} + 1.9R \cdot \mathbb{1}_{\{D_1 > D_b\}}, & \text{for } M_1 > M_2 \\ 1.8R \cdot \mathbb{1}_{\{D_2 \leq D_b\}} + 2R \cdot \mathbb{1}_{\{D_2 > D_b\}}, & \text{for } M_1 < M_2 \end{cases}, \\ h(R) &= \begin{cases} 0.5R \cdot \mathbb{1}_{\{D_1 \leq D_c\}} + 0.6R \cdot \mathbb{1}_{\{D_1 > D_c\}}, & \text{for } M_1 > M_2 \\ R \cdot \mathbb{1}_{\{D_2 \leq D_c\}} + 1.1R \cdot \mathbb{1}_{\{D_2 > D_c\}}, & \text{for } M_1 < M_2 \end{cases}, \\ \phi(K) &= \begin{cases} 0.8K \cdot \mathbb{1}_{\{D_1 \leq D_d\}} + 0.85K \cdot \mathbb{1}_{\{D_1 > D_d\}}, & \text{for } M_1 > M_2 \\ 0.95K \cdot \mathbb{1}_{\{D_2 \leq D_d\}} + 0.98K \cdot \mathbb{1}_{\{D_2 > D_d\}}, & \text{for } M_1 < M_2 \end{cases}, \\ \gamma(K) &= \begin{cases} 0.55K \cdot \mathbb{1}_{\{D_1 \leq D_e\}} + 0.6K \cdot \mathbb{1}_{\{D_1 > D_e\}}, & \text{for } M_1 > M_2 \\ 0.7K \cdot \mathbb{1}_{\{D_2 \leq D_e\}} + 0.75K \cdot \mathbb{1}_{\{D_2 > D_e\}}, & \text{for } M_1 < M_2 \end{cases}, \\ \text{and } \eta(K) &= \begin{cases} 0.2K, & \text{for } M_1 > M_2 \\ 0.5K, & \text{for } M_1 < M_2 \end{cases}. \end{aligned}$$

Deutsche Bank Securities led a new transformer reinsurance deal, and Embarcadero sold \$150 million 3-years catastrophe bonds at a rate of 7.25 percent above three-month U.S Treasury bills in 2012 compared to 6.6 percent in 2011 [26]. Therefore, it is realistic to assume an 3% – 4% coupon rate if no major earthquakes occurs plus a 3% premium ².

In the one-period case, we assume that K , r , π , R and e are constant. Therefore, cash flow is independent to financial risks, we could apply Eq. (10), and obtain the price of the CAT bond

$$V(d) = \frac{1}{1 + (r + e)} \cdot \frac{1}{1 + \pi} \mathbb{E}^{\mathbb{P}} [d(M, R, D_q)], \quad (11)$$

where \mathbb{P} is the probability measure corresponding to the distribution of M_1 , M_2 , D_1 and D_2 . It is important to note that one of our financial market influence rate $(r + e)$ is a shift of interest rate which makes the CAT bonds more attractive than normal return bonds.

Assuming that expectation in (11) exists, one can approximate the CAT bond price by the same line logic of Zimbidis *et al.* [61] by using equilibrium pricing theory,

$$V(d) = \lim_{h \rightarrow \infty} V(d)^{(h)},$$

where

$$V(d)^{(h)} = \frac{1}{1 + (r + e)} \frac{1}{1 + \pi} \frac{1}{h} \sum_{l=1}^h d(M^{(l)}, R^{(l)}, D_q^{(l)}), \quad (12)$$

Therefore, the real value $V(d)$ can be calculated by Monte Carlo method, where h represent the simulations number, Boyle *et al.* [5] and Romaniuk [49], we will address this as an application in Section 4.

3.2 Multi Period (advanced) Model

Under the discrete time framework, we now introduce the symbols and notation of multi period models. K , e , D_q and the coupon payment functions (i.e. $f(R)$, $g(R)$, $h(R)$, $\phi(K)$, $\gamma(K)$, and $\eta(K)$) have the same meaning as in the one period model.

T : is the maturity date of the CAT bond.

r_k : is market yield on 1-year US Treasury securities rate at time k, more precisely, r_k gives the annual compounded interest discount rate of a typical cash flow for the period k+1. We assume that r_k follows Autoregressive Integrated Moving Average ARIMA(1, 1, 1) model [3] with parameters θ_1 and α_1 for any $k = 1, 2, \dots, T$ (that assumption coincides with the practical experience), and note that $r_k > 0$,

$$\Delta r_k = C_1 + \theta_1 \Delta r_{k-1} + \varepsilon_k + \alpha_1 \varepsilon_{k-1},$$

where $\Delta r_k = r_k - r_{k-1}$, C_1 is constant and the error terms ε_k are generally assumed to be independent, identically distributed variables sampled from a normal distribution with zero mean.

² More information for numerical analysis will be available in Section 4.

π_k : is the one year inflation rate at time k. Similar setting as Treasury rate, we assume that $\pi_k (\pi_k > 0)$ follows ARIMA(1, 0, 0) model with parameters α_2 for any $k = 1, 2, \dots, T$, $\pi_k > 0$,

$$\pi_k = C_2 + \varepsilon_k + \alpha_2 \varepsilon_{k-1}.$$

R_k : is the 12-months LIBOR issuing at time k. Due to popularity, we assume the fundamental process in instantaneous LIBOR rate $\{R_k; k \in [0, T]\}$ modelling is the CIR process [13] given by the following stochastic differential equation,

$$dR_k = \alpha_3(\mu_3 - R_k)dt + \sqrt{R_k}\sigma_3 dW_t, \quad (13)$$

where $\theta_3 = (\alpha_3, \mu_3, \sigma_3)$ are model parameters and W_t is the standard Brownian motion. The LIBOR rate process R_k stays on a positive domain which is guaranteed by diffusion function $R_k\sigma_3^2$.

M_k : is the annual maximum magnitude level of the earthquake within specified regions of California in the k^{th} year, $M_k = \max\{(M_1)_k, (M_2)_k\}$, for $k = 1, 2, \dots, T$, where $(M_1)_k$ and $(M_2)_k$ have the common distributions described in Section 3.3.1.

$d(M_k, R_k, D_q)$: is the cash value received by bondholder of the CAT bond at time $k = 1, 2, \dots, T$, constructed by the following form

$$\begin{aligned} d(M_k, R_k, D_q) = & \begin{cases} Kf(R_k)\mathbb{1}_{\{0 \leq M_k \leq M_a\}} + Kg(R_k)\mathbb{1}_{\{M_a < M_k \leq M_b\}} + Kh(R_k)\mathbb{1}_{\{M_b < M_k \leq M_c\}}, \\ \text{for } k = 1, 2, \dots, T-1 \\ K(1 + f(R_k))\mathbb{1}_{\{0 \leq M_k \leq M_a\}} + K(1 + g(R_k))\mathbb{1}_{\{M_a < M_k \leq M_b\}} + \\ + K(1 + h(R_k))\mathbb{1}_{\{M_b < M_k \leq M_c\}} + K\mathbb{1}_{\{M_c < M_k \leq M_d\}} + \phi(K)\mathbb{1}_{\{M_d < M_k \leq M_e\}} + \\ + \gamma(K)\mathbb{1}_{\{M_e < M_k \leq M_f\}} + \eta(K)\mathbb{1}_{\{M_k > M_f\}}, \text{ for } k = T \end{cases} \end{aligned} \quad (14)$$

Therefore,

$$\begin{aligned} \bar{d}(M_k, R_k, D_q) = \mathbb{E}^{\mathbb{Q}}[d(k)|\mathcal{A}_k^{(1)}] = & \begin{cases} \mathbb{E}^{\mathbb{Q}} [Kf(R_k)\mathbb{1}_{\{0 \leq M_k \leq M_a\}} + Kg(R_k)\mathbb{1}_{\{M_a < M_k \leq M_b\}} + Kh(R_k)\mathbb{1}_{\{M_b < M_k \leq M_c\}}] \\ \text{for } k = 1, 2, \dots, T-1 \\ \mathbb{E}^{\mathbb{Q}}[K(1 + f(R_k))\mathbb{1}_{\{0 \leq M_k \leq M_a\}} + K(1 + g(R_k))\mathbb{1}_{\{M_a < M_k \leq M_b\}} + \\ + K(1 + h(R_k))\mathbb{1}_{\{M_b < M_k \leq M_c\}} + K\mathbb{1}_{\{M_c < M_k \leq M_d\}} + \phi(K)\mathbb{1}_{\{M_d < M_k \leq M_e\}} + \\ + \gamma(K)\mathbb{1}_{\{M_e < M_k \leq M_f\}} + \eta(K)\mathbb{1}_{\{M_k > M_f\}}], \text{ for } k = T \end{cases} \end{aligned}$$

Trivially, cash flows from this multi-period CAT bond depend on both financial and catastrophic risk variables. Therefore, according to Eq (9) price of T-period the CAT bond is:

$$V(d) = \mathbb{E}^{\mathbb{Q}_1} \left[\sum_{k=1}^T \frac{1}{\prod_{s=0}^{k-1} [1 + r(s) + e][1 + \pi(s)]} \bar{d}(M_k, R_k, D_q) \right], \quad (15)$$

which could be calculated by the same method as for Eq (12). Assuming that expectation in (15) exists, similar as one period model, CAT bond price can be approximated by Strong Law of Large Numbers,

$$V(d) = \lim_{h \rightarrow \infty} V(d)^{(h)},$$

where

$$V(d)^{(h)} = \frac{1}{h} \sum_{l=1}^h \sum_{k=1}^T \frac{1}{\prod_{s=0}^{k-1} [1 + r^{(l)}(s) + e][1 + \pi^{(l)}(s)]} \bar{d}(M_k^{(l)}, R_k^{(l)}, D_q^{(l)}), \quad (16)$$

where h represent the number of simulation, Zimbidis *et al.* [61], Boyle *et al.* [5] and Romaniuk [49]. Detail numerical example for Monte Carlo simulation will be shown in Section 4.

For future convenient, we employed the trigger points of magnitude and depth as $M_a = 5.4$, $M_b = 5.8$, $M_c = 6.2$, $M_d = 6.6$, $M_e = 7.0$, $M_f = 7.4$, and $D_a = 20$, $D_b = 15$, $D_c = 10$, $D_d = 10$, $D_e = 10$. A catastrophe might or might not occur prior to the maturity date T . According to the cash flow stream given in Eq. (14), the CAT bond with the face amount of $\$K$ will pay coupons to bondholders $f(R)$, $g(R)$ and $h(R)$ at the end of each period if the maximum magnitude earthquake that occurred in this period is between $(0, 5.4]$, $(5.4, 5.8]$, $(5.8, 6.2]$, respectively or no coupon payment if the magnitude level is larger than 6.2. At the maturity date, the CAT bond is scheduled to repay the full principle payment plus a coupon, a $\phi(K)$, a $\gamma(K)$, and a $\eta(K)$, if the magnitude is between $(0, 6.6]$, $(6.6, 7.0]$, $(7.0, 7.4]$ and $(7.4, \infty)$, respectively. Note in this stepwise parameterized model, the region q and the corresponding depth of the earthquakes should also be evaluated.

3.3 Applications Of The California Earthquake Data For Catastrophic Risk Variables

In this subsection we are going to select the earthquake data in California as an example and to estimate the distributions of the magnitude and the depth for the future time period.

Figure 1 displays the recent significant earthquakes in California and the darker color represents the more severe ones. We have produced two circles where the most significant earthquakes occurred. Thus, we will analyse the earthquakes that have hit the circled areas - which include the city of San Francisco (Region 1) and Los Angeles (Region 2) - over the period 1968 to 2011. Table 1 (data from Southern California Earthquake Data Center [52]) illustrates the series of annual maximum magnitude earthquakes in each region, and the latitude, longitude and depth of the corresponding earthquake. Actually, these two regions which include the biggest cities in California claim the majority of the economic losses.

Table 1 Annual Maximum Earthquakes In Two Regions In California

Y	Region 1					Region 2					Y	Region 1					Region 2				
	La	Lo	D	M		La	Lo	D	M			La	Lo	D	M		La	Lo	D	M	
1968	37.0	-121.7	12	4.3		33.2	-116.1	10	6.6		1990	36.9	-121.7	6	5.4		34.1	-117.7	4	5.5	
1969	38.5	-122.8	5	5.7		33.3	-116.4	6	5.5		1991	35.8	-121.3	10	5.2		34.3	-118.0	9	5.8	
1970	36.8	-121.4	10	4.7		34.3	-117.5	11	5.2		1992	38.9	-122.8	4	4.5		34.2	-116.4	1	7.3	
1971	36.6	-121.2	4	4.7		34.4	-118.4	9	5.8		1993	37.0	-121.5	8	4.8		35.1	-119.1	21	5.2	
1972	36.6	-121.2	4	5.1		34.8	-118.9	4	4.3		1994	38.8	-122.4	12	4.2		34.2	-118.5	18	6.7	
1973	39.3	-123.4	18	4.8		34.1	-119.0	8	5.3		1995	36.6	-121.2	7	4.9		34.4	-118.7	13	5.0	
1974	36.9	-121.5	5	5.2		34.4	-118.5	24	4.7		1996	36.9	-121.6	5	4.9		34.0	-116.3	6	4.5	
1975	39.4	-121.5	5	5.7		34.5	-116.5	0	5.3		1997	39.9	-120.8	5	4.3		35.0	-116.8	2	5.3	
1976	36.5	-120.5	12	4.6		33.1	-115.6	5	5.1		1998	36.8	-121.5	9	5.1		34.3	-116.8	6	4.8	
1977	39.4	-123.3	15	4.8		34.4	-118.5	11	4.4		1999	37.9	-122.7	8	4.6		34.6	-116.3	0	7.1	
1978	39.2	-123.2	12	4.5		34.3	-119.7	18	5.1		2000	38.4	-122.4	10	4.9		32.9	-115.5	10	4.5	
1979	37.1	-121.5	9	5.8		32.7	-115.4	15	5.8		2001	39.8	-120.6	18	5.3		33.5	-116.5	15	5.0	
1980	37.8	-121.8	15	5.8		33.5	-116.5	19	5.3		2002	37.0	-121.6	7	4.9		34.8	-116.3	5	4.8	
1981	37.4	-121.7	9	4.8		33.1	-115.6	19	5.8		2003	35.7	-121.1	9	6.5		34.3	-116.8	1	5.0	
1982	36.6	-121.3	11	4.6		33.5	-116.7	13	4.8		2004	36.0	-120.5	11	4.9		35.4	-118.6	4	5.0	
1983	35.8	-121.4	10	5.2		32.5	-117.4	10	4.7		2005	38.8	-122.8	0	4.3		33.5	-116.6	14	5.2	
1984	37.3	-121.7	9	6.2		34.0	-119.0	13	4.6		2006	38.8	-122.8	5	4.7		32.7	-116.0	14	4.6	
1985	39.5	-120.6	7	4.6		35.5	-118.9	6	4.6		2007	37.4	-121.8	10	5.5		33.7	-117.5	13	4.7	
1986	37.5	-121.7	9	5.7		34.0	-116.6	10	5.7		2008	40.0	-120.9	0	4.5		34.0	-117.8	15	5.4	
1987	36.2	-120.7	28	4.8		33.0	-115.9	11	6.6		2009	35.6	-121.0	6	4.4		33.3	-115.7	6	4.8	
1988	37.4	-121.7	10	5.3		34.9	-118.7	7	5.4		2010	37.5	-121.8	11	4.1		32.7	-115.9	5	5.7	
1989	37.0	-121.9	17	7.0		33.9	-118.6	12	5.0		2011	36.6	-121.2	8	4.6		32.7	-115.7	15	4.4	

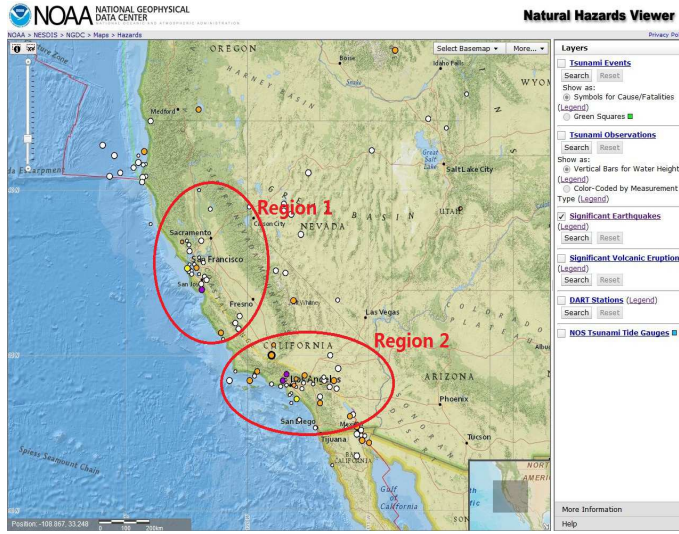


Fig. 1 Recent Significant Earthquakes in the Region of California (Source: NOAA National Geophysical Data Center)

3.3.1 Magnitude

The traditional approach to define extremes is to focus on the statistical behavior of

$$M_k = \max\{X_{1k}, X_{2k}, \dots, X_{ok}\},$$

where $X_{1k}, X_{2k}, \dots, X_{ok}$ is a sequence of $o = 365$ independent random variables having a universal distribution function F which measures the daily maximum magnitude of earthquakes in the areas under consideration for the period of $[k, k + 1)$. Thus, the sequence of M_k corresponds to the k^{th} annual maximum magnitude over the observation period 1968 to 2011 (see Table 1). The distribution of M_k can be derived for all values of k by the Generalized Extreme Value Distribution (GEM). The re-scaled sample maxima $M_k^* = (M_k - b_k)/a_k$ is a heavy tail distribution and the possible distribution is provided by the well-known Theorem of Fisher-Tippet, Gnedenko (Fisher and Tippet [19]; Gnedenko [28]; Embrechts, *et al.*: Theorem 3.2.3 [18] and Coles: Theorem 3.1 [10]).

Theorem 4 (Fisher-Tippet, Gnedenko)

If there exists sequences of constants $\{a_k : k > 0\}$ and $\{b_k : k > 0\}$ such that

$$\mathbb{P} \left\{ \frac{M_k - b_k}{a_k} \leq z \right\} \rightarrow G(z) \quad \text{as } k \rightarrow \infty,$$

for a non-degenerate distribution function G , then G is a member of the GEV family

$$G(z) = \exp \left\{ - \left[1 + \xi \left(\frac{z - \mu}{\sigma} \right) \right]^{-1/\xi} \right\}, \quad (17)$$

defined on $\{z : 1 + \xi(z - \mu)/\sigma > 0\}$, where $-\infty < \mu < \infty$, $\sigma > 0$ and $-\infty < \xi < \infty$.

The model has three parameters which are: location parameter μ , scale parameter σ , and shape parameter ξ . When $\xi = 0$ as the limit of Eq. (17) as $\xi \rightarrow 0$, the model corresponds to the Gumbel family. For the case $\xi > 0$ and $\xi < 0$, the Eq. (17) leads to Fréchet, and Weibull family distribution, respectively. Then, the GEV parameters can be estimated by maximizing the log-likelihood function, see approach conducted by Zimbidis *et al.* [61].

The time series plots of the maxima for both regions (Figure 2) suggest the data as independent observations from the GEV distribution, assuming that the pattern of variation have stayed constant over the observed period.

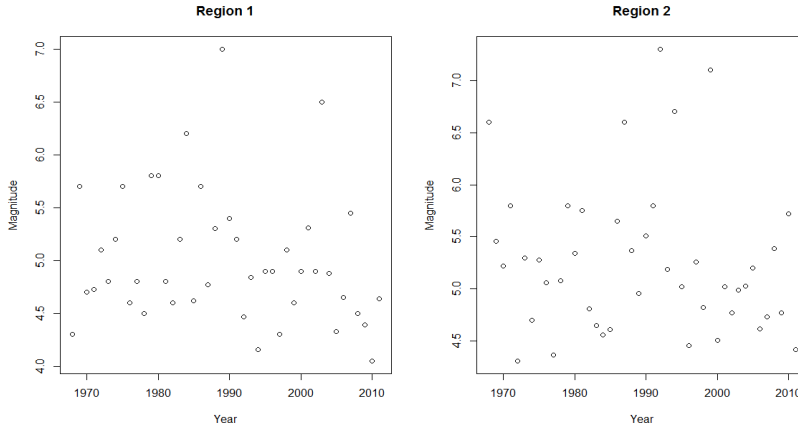


Fig. 2 Scatter Plot of the Annual Maximum Magnitude Earthquakes of Region 1 (left) and Region 2 (right) in California.

We will take the Region 1 as an example of analysis. Then, produce the maximization of the GEV log-likelihood for these data and achieve the estimated parameter

$$(\hat{\mu}, \hat{\sigma}, \hat{\xi}) = (4.71946946, 0.44861472, 0.05866229),$$

for which the log-likelihood is 36.01543. The approximate variance-covariance matrix of the parameter estimates is

$$V = \begin{bmatrix} 0.005854675 & 0.001935385 & -0.003127097 \\ 0.001935385 & 0.003228341 & -0.001542433 \\ -0.003127097 & -0.001542433 & 0.013764031 \end{bmatrix}.$$

Therefore, we can easily obtain the standard errors 0.07651585, 0.05681849 and 0.11732021 for μ , σ and ξ respectively, while the approximate 95% confident intervals for each parameter are $\mu \in [4.64, 4.80]$, $\sigma \in [0.39, 0.51]$ and $\xi \in [-0.06, 0.18]$.

In order to check the fitness of the GEV model, we construct the various diagnostic plots of annual maximum earthquakes in Region 1 of California data in Figure 3. The probability plot and the quantile plot are close linear which support the validity of the fitted model. The estimate of ξ is close to zero, which displayed in the return level plot that estimated curve is near-linear. Based on the histogram density plot of the data, the density estimate is consistent. Consequently, the above analysis gives strong evidence for the fitted GEV model.

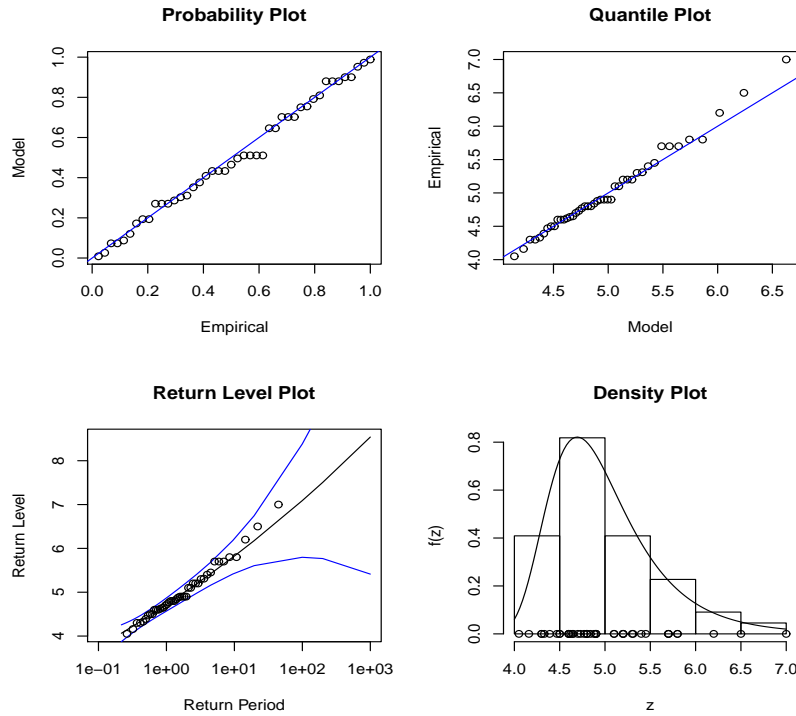


Fig. 3 Diagnostic plots for GEV fit to the Annual Maximum Earthquakes of Region 1 California.

Furthermore, the tail behavior³ of the distribution displayed in Figure 4 is the plot of the sample mean excess function, and the downward trend suggests a very short tail behavior for the annual maximum earthquakes of Region 1 of California [61].

Similar analysis can be conducted for Region 2, and the exceeding probabilities intervals in Region 1 and 2 of California for the generalized extreme value distributions are illustrated in Table 2. It is easy to conclude that this CAT bond is very attractive to investors since the possibility of an earthquake occurring in target re-

³ More information also see Beirlant *et al.* [4]; Embrechts *et al.* [18] and others.

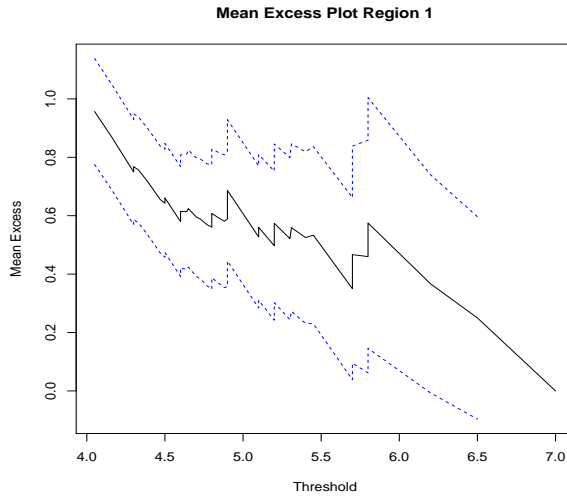


Fig. 4 A sample mean excess plot of the Annual Maximum Earthquakes of Region 1 California with 95% confidence interval.

gions with magnitude larger than 6.6 is less than 8%, which is to say, we introduce a bond with 92% capital guarantee.

Table 2 Exceeds Probabilities For The Model In Region 1 and 2 California

	Region 1	Region 2
$\mathbb{P}(5.0 < M \leq 5.4)$	0.205599827	0.250286661
$\mathbb{P}(5.4 < M \leq 5.8)$	0.105005438	0.150399888
$\mathbb{P}(5.8 < M \leq 6.2)$	0.049947001	0.080828401
$\mathbb{P}(6.2 < M \leq 6.6)$	0.023619037	0.042623696
$\mathbb{P}(6.6 < M \leq 7.0)$	0.011371076	0.022816466
$\mathbb{P}(7.0 < M \leq 7.4)$	0.005618113	0.012543028
$\mathbb{P}(M > 7.4)$	0.006178647	0.01813135

3.3.2 Depth

The next stage is to analyze the depth distribution of corresponding earthquakes in both regions shown in Table 1. According to the density plot Figure 5, the depth of the earthquake is a right skewed heavy tail distribution and we fit it as a gamma distribution

$$f(x; \alpha, \beta) = \beta^\alpha \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x},$$

with estimated parameters are $(\hat{\alpha}, \hat{\beta}) = (2.35378504, 0.25460951)$ and $(\hat{\alpha}, \hat{\beta}) = (1.44878306, 0.14585340)$, for Region 1 and 2, respectively. This model is realistic since the earthquakes which occur near the surface tend to have higher magnitude compared with the deeper ones [24].

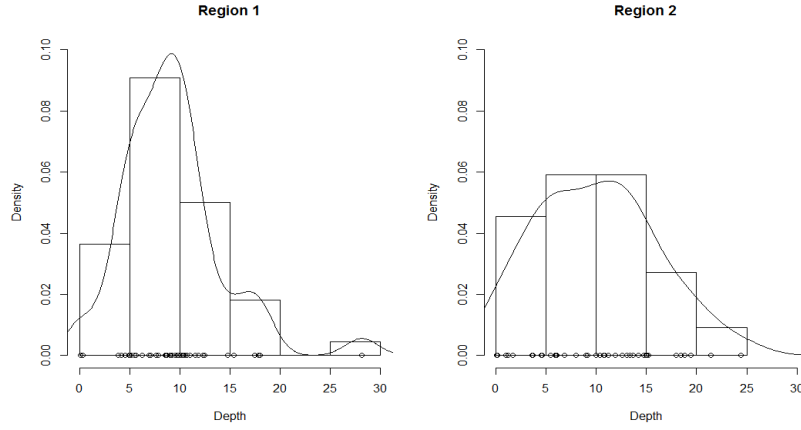


Fig. 5 Density Plot for the Depth of Annual Maximum Earthquakes in Region 1 and Region 2.

4 Numerical Example and Simulation

4.1 Numerical Example for the one-period model

Assuming the one period model with face value $\$K = 1000$, $r = 0.12\%$ and $\pi = 3.16\%$ for one year interest rate and inflation rate respectively (data from Board of Governors of the Federal Reserve System [2]). Given risk premium $e = 3\%$ and Libor rate $R = 1.13\%$ (on the day 30/12/2011 [34]), according to Eq (12) and obtain the price of one period CAT bond $\$V = 940$.

4.2 Simulation and Pricing for the multi-period model

We consider a 5-year period CAT bond with payments depending on the magnitude and depth of earthquakes in the set regions as described in section 3.1. According to the fact that the probability of large magnitude earthquakes occur are low, large number of simulation estimates price of CAT bond with relatively small error, Romaniuk [49]. We build the simulation using the following 5 steps.

Step 1:

Firstly, the maximum magnitude of the earthquake in each region can generate 100,000 sequence values by GEV distributions representing the 5-year period up

to the maturity date. Moreover, we can generate 100,000 sequences of depths for both regions in Gamma distribution. The distribution of the annual maximum magnitude earthquakes and the depth have been evaluated in the previous section. Then, we compare the corresponding sequence of 100,000 magnitude in both regions, and choose the larger magnitude branch for the future simulation.

Step 2:

Secondly, we obtain 100,000 different paths for the LIBOR rate R_k , $k \in [0, 5]$ using Monte Carlo simulations. Following the description of Romaniuk [49], we use Iterative Stochastic Equation by the concept of local characterizations for the Levy process.

In our simulation, we let $[0, T]$ be the life time interval for the CAT bond and discrete it to δ different steps. The time moments are $\tau = \{\tau_0 = 0, \tau_1, \dots, \tau_\delta = T\}$, and δ is the number of steps. The steps are constant as one day (250 business days a year), $\Delta\tau = \tau_{v+1} - \tau_v$, here $v = 1, 2, \dots, \delta - 1$. The discrete version of (13) given by Kladvko [33] is the form

$$R_{\tau+\Delta\tau} - R_\tau = \alpha_3(\mu_3 - R_\tau)\Delta\tau + \sigma_3\sqrt{R_\tau}\sqrt{\Delta\tau}\varepsilon_\tau, \quad (18)$$

where ε_τ follow $N(0, \Delta\tau)$ as a white noise process for $\tau = 1, 2, \dots$.

The MATLAB implementation of the estimation processes provided by Kladvko [33] suggest to use Ordinary Least Square of Eq (18) to find starting point of parameters. And then maximize the log-likelihood function of the CIR process. Therefore, the statistical analysis of 12-months LIBOR historical data [34] (2000–2011) obtains the estimated parameters are $\hat{\theta} = (\hat{\alpha}_3, \hat{\mu}_3, \hat{\sigma}_3) = (0.212421, 1.084655, 0.420791)$. For the initial value R_0 of Eq (18) we set $R_0 = 1.13\%$ which is the actual LIBOR rate value in December 2011.

Step 3:

The next step is the generation of sequences for the annual interest and inflation rate (data from Board of Governors of Federal Reserve System for the period 1968 to 2011 [2]). Recall back to section 3.1, r_k follows ARIMA(1, 1, 1) model with parameters $(\hat{C}_1, \hat{\theta}_1, \hat{\alpha}_1) = (-0.0976, -0.2833, 1)$ and π_k follows ARIMA(1, 0, 0) model with parameters $(\hat{C}_2, \hat{\alpha}_2) = (0.7867, 0.7867)$, for any $k = 1, 2, 3, \dots$, $r_k \geq 0$ and $\pi_k \geq 0$, according to Maximum Log-Likelihood estimation of 1-year US Treasury securities rate and inflation rate from 1968 to 2011.

Step 4:

The following step is to calculate the coupon payments (cash flows $d(k)$) of the CAT bond for the five year period. It should be mentioned that this procedure is quite complex and involves logical functions and many subroutines. According to the cash flow stream in Eq. (14), our CAT bond may diminish capital if and only if a magnitude level above 6.6 earthquake hits California before the maturity date. It is attractive for conservative investors because the possibility of losing capital is less than 8%, see Table 2. Moreover, we assume a value for the face amount $\$K = 1000$ and a certain value risk premium $e = 3\%$.

Step 5:

The final step is the calculation of the present value of cash flows for every year, and average over all the discounted values based on the r_k, π_k for each period. According to Eq. (16), the price of the $T = 5$ CAT bond approximately equals to $\$V = 779.73$.

Now we are going to find the validity of results. In the above simulation, we are using the equilibrium pricing theory given in Eq. (16) for $h = 100,000$, and we run the algorithm 100 times to generate 100 possible value of this CAT bond, and variance equals to 0.91. It could be easily derived that the variances of price drop dramatically as the increasing of the number of h , and asymptotically equal to zero after 10,000. Figure 6 is the density plot of prices value where the density reach the mode at 778.62 with the value 0.43. This is a quite promising result since the low volatility level suggests our pricing model is both consistent and computational efficient.

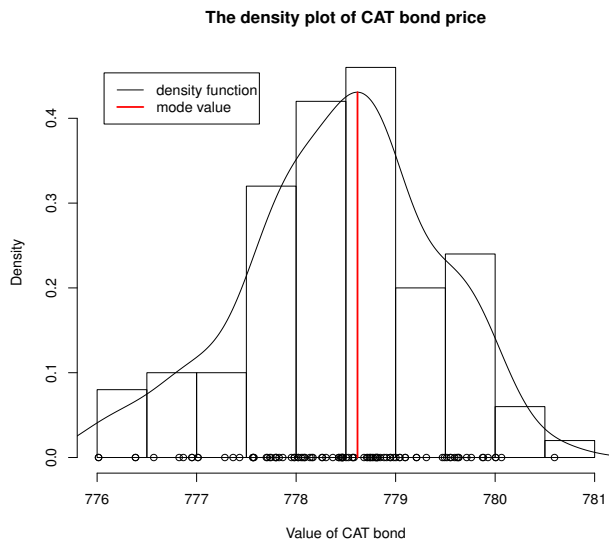


Fig. 6 The Density Plot(Left) and the Cumulated Density Plot(Right) of the CAT Bond Price by Running the Algorithm 100 Times with the Constant Value $h = 100,000$

5 Concluding Remarks

In this paper, we have built a valuation framework for earthquake CAT bonds with n financial and m catastrophic independent risks following a generalized framework as in Cox and Perdersen [12]. These securitization products can play a vital role for both insurers and reinsurers sustainability in the financial environment, as well as for the governmental authorities, since they have noticed the increasing importance of spreading of the catastrophe risk. The high return of the CAT bond that have been produced in this paper can generate sufficient funds to repay the claims and post-disaster reconstruction, if a significant catastrophe event attacks the area. Furthermore, the assumptions made in this paper are quite standard and realistic which make the valuation model easy to modify further and apply in the industry. To simplify the model, we have limited all the risks to be pairwise independent. It is quite natural that

an earthquake occurs in certain regions, and generally speaking it would not affect the whole world exchange and production level and economic environment.

Furthermore, we have demonstrated how to contract a practical pricing model for the earthquakes in California from 1968 to 2011. We have employed Extreme Value Theory for the maximum magnitude of the earthquakes each year and conclude that they follow Frechét distribution in our case. In addition, the depth of those earthquakes have fitted to a Gamma distribution. For the financial risks, we have chosen the classical ARIMA model for interest and inflation rate, and CIR model for the stochastic process of coupon payment as a predetermined function of annual Libor rate. Consequently, using the Monte Carlo simulation method, we have produced an equilibrium price of the earthquake CAT bond depending on the risk variables above. This model, as an extension of Cox and Pedersen [12], provides a more accurate approximation of price by considering multi-variables cross financial and catastrophe risks. However, with the appearance of catastrophe risks, CAT bonds cannot be perfectly hedged in the incomplete market and this high yields received may insufficient to bare the risks for the investors.

There is substantial literature in this area for alternative approaches, which can be considered in the future. One possible extension based on the argument of dependency is to produce joint distributions of the financial risks variables or to deal with the dependency of earthquake side effects, i.e. tsunami and out break of fires during an earthquake. Another direction is to look at the more complex problem relating other catastrophic events such as windstorm and terrorist attack.

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